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TORSION OF A SPHERICAL LAYER BY A SPHERICAL ANNULAR STAMP*

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Methods of reducing dual series equations to infinite algebraic systems /1,2/ are used to study a mixed problem of the theory of elasticity concerning the torsion of a spherical layer by a spherical annular stamp. The inner or outer surface of this layer is rigidly clamped, and the stamps are coupled to the other surface of the layer. The resulting infinite algebraic systems of first kind are reduced, after the regularization, to the systems of second kind, and the latter can be solved using the method of consecutive approximations. Authors of /3,4/ used the same method to study certain dynamic problems of torsion of bodies with spherical surfaces.

1. Let us consider a problem of torsion of an elastic, spherical layer with one of the surfaces $r = R_0$ rigidly clamped, the torsion caused by rotation about the axis $\theta = 0$ by the angle ε, of an arbitrary, spherical (circular or annular) stamp attached to the other surface $r = R$ of the layer along the segment $\theta_1 \leqslant \theta \leqslant \theta_2$. The directions of the axes of the spherical (r, θ, φ)-coordinate system is chosen such that $\theta_1 < \pi/2$ and $\theta_2 \neq \pi$. When $\theta_1 = 0$, the spherical stamp becomes circular, and $\theta_1 \neq 0$ corresponds to the annular stamp. We assume that the surface $r = R$ of the spherical layer is stress free outside the stamp.

Let us introduce special coordinates (t, x, φ) connected to the spherical (r, θ, φ)-coordinates by the relations

$$t = \ln (rR^{-1}), \quad x = \cos\theta \tag{1.1}$$

We seek to determine a displacement function $\psi(t, x)$ satisfying the following differential equation /5/ and boundary conditions:

$$\frac{\partial^2\psi}{\partial t^2} + (1-x^2)\frac{\partial^2\psi}{\partial x^2} + 3\frac{\partial\psi}{\partial t} - 4x\frac{\partial\psi}{\partial x} = 0 \tag{1.2}$$

$$\psi(a, x) = 0, \quad -1 \leqslant x \leqslant 1 \tag{1.3}$$

$$\psi(0, x) = \varepsilon, \quad b \leqslant x \leqslant c$$

$$\sqrt{1-x^2}\,\frac{\partial\psi(t,x)}{\partial t}\Big|_{t=0} = 0, \quad -1 \leqslant x < b, \; c < x \leqslant -1$$

$$(a = \ln(R_0R^{-1}), \quad b = \cos\theta_2, \quad c = \cos\theta_1)$$

The displacements u_φ and stresses $\tau_{t\varphi}$ and $\tau_{x\varphi}$ are connected with the displacement function by means of the relations

$$u_\varphi = Re^t\sqrt{1-x^2}\,\psi$$
$$\tau_{t\varphi} = G\sqrt{1-x^2}\,\partial\psi/\partial t, \quad \tau_{x\varphi} = G(1-x^2)\,\partial\psi/\partial x$$

The function /5/

$$\psi(t, x) = e^{-3/2t}\sum_{k=1}^{\infty} A_k\left[\operatorname{sh}\left(\frac{2k+1}{2}t\right) - \operatorname{th}\left(\frac{2k+1}{2}a\right)\operatorname{ch}\left(\frac{2k+1}{2}t\right)\right]\frac{d}{dx}P_k^1(x) \tag{1.4}$$

is a solution of the differential equation (1.2) satisfying the first boundary condition of (1.3). In order for the function $\psi(t, x)$, give.. by (1.4), to be a solution of the problem in question, the coefficients A_k must satisfy, in accordance with the last two boundary conditions of (1.3), the dual series equations which are written in the form

$$\sum_{k=1}^{\infty} B_kK(u_k)\,y(u_k, x) = P_1^1(x), \quad b \leqslant x \leqslant c \tag{1.5}$$

$$\sum_{k=1}^{\infty} B_k y(u_k, x) = 0, \quad -1 \leqslant x < b, \quad c < x \leqslant 1$$

where

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$$B_k = -\frac{1}{\varepsilon}\sqrt{\frac{u_k^2 - {}^1/_4}{u_k}}\left[u_k + \frac{3}{2}\,\mathrm{th}\,(au_k)\right]A_k \qquad (1.6)$$

$$y(u_k, x) = \sqrt{\frac{u_k}{u_k^2 - {}^1/_4}}\,P^1_{-{}^1/_2+u_k}(x), \quad u_k = k + \frac{1}{2}$$

$$K(u) = \frac{\mathrm{sh}\,(au)}{{}^3/_2\,\mathrm{sh}\,(au) + u\,\mathrm{ch}\,(au)}$$

($P_k^1(x)$ is a first order Legendre function).

The functions $y(u_k, x)$ are eigenfunctions (corresponding to the eigenvalues u_k) of the Sturm—Liouville problem

$$Ly = u^2 y, \quad Ly = -(1 - x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + \left(\frac{1}{4} + \frac{1}{1 - x^2}\right)y \qquad (1.7)$$

$$y(u, -1) = 0, \quad y(u, 1) = 0$$

The functions $y(u_k, x)$ $(k = 1, 2, \ldots)$ form a complete orthonormal system of functions on the segment /-1,1/

$$\int_{-1}^{1} y(u_k, x)\,y(u_n, x)\,dx = \begin{cases} 0, & n \neq k \\ 1, & n = k \end{cases} \qquad (1.8)$$

The function $K(u)$ of (1.6) is an even meromorphic function in the complex $u = s + i\sigma$ variable plane. All its zeros $\pm i\delta_n$ and poles $\pm i\gamma_n$ $(n = 1, 2, \ldots)$ are imaginary. Let us arrange the zeros $z = i\delta_n$ and poles $\zeta = i\gamma_n$ lying in the half-plane $\sigma > 0$ into a sequence in the order of increasing moduli. We see that

$$\delta_n = \frac{\pi n}{|a|}, \quad \frac{(2n - 1)\pi}{2|a|} < \gamma_n < \frac{\pi n}{|a|}$$

$$\gamma_n \to \frac{(2n - 1)\pi}{2|a|} \ \text{for}\ n \to \infty$$

Taking into account the fact that the estimate

$$K(u) = O(u^{-1}), \quad |u| \to \infty$$

holds on any correct system of contours Γ_n lying in the $u = s + i\sigma$ -plane, we can write the function $K(u)$ in the form

$$K(u) = A\frac{N_1(u^2)}{N_2(u^2)} = A\prod_{n=1}^{\infty}\frac{1 + u^2\delta_n^{-2}}{1 + u^2\gamma_n^{-2}} \qquad (1.9)$$

$$A = \frac{2a}{2 + 3a}$$

where ($N_1(u^2)$ and $N_2(u^2)$ are entire functions.

We define

$$q(x) = \sum_{k=1}^{\infty} B_k y(u_k, x) \qquad (1.10)$$

noting that the function $q(x)$ (1.10) coincides, with the accuracy of up to the factor $G\varepsilon$, with the stress distribution function $\tau_{t\varphi}(0, x)$.

Taking into account (1.9), (1.10) and the fact that the operator L (1.7) transforms the function $y(u_k, x)$ into the function $u_k^2 y(u_k, x)$, we can write the first relation of the dual equation (1.5) in the form

$$AN_1(L)q(x) = N_2(L)P_1^1(x) \qquad (1.11)$$

where $N_1(L)$ and $N_2(L)$ denote the differential operators in x, of infinite order. A solution of the differential equation (1.11) with respect to the function $q(x)$ has the form

$$q(x) = K^{-1}(u_1)P_1^1(x) + \sum_{n=1}^{\infty} H_n(x), \quad b \leqslant x \leqslant c \qquad (1.12)$$

$$H_n(x) = C_n^\circ P^1_{-{}^1/_2+i\delta_n}(x) + C_n^* P^1_{-{}^1/_2-i\delta_n}(x) + D_n^\circ Q^1_{-{}^1/_2+i\delta_n}(x) + D_n^* Q^1_{-{}^1/_2-i\delta_n}(x) \qquad (1.13)$$

The first term in (1.12) represents a particular solution of the inhomogeneous equation which can be found using the symbolic method, and the infinite sum gives a general solution of the homogeneous equation. The associated Legendre functions $P^1_{-{}^1/_2+u}(x)$ and $Q^1_{-{}^1/_2+u}(x)$ represent the linearly independent solutions of (1.7), and C_n°, C_n^*, D_n°, D_n^* are constants.

Using the functional relations /6/

$$P^1_{-{}^1/_2+u}(-x) = P^1_{-{}^1/_2+u}(x)\cos[({}^1/_2 + u)\pi] - 2/\pi\, Q^1_{-{}^1/_2+u}(x)\sin[({}^1/_2 + u)\pi]$$

$$P^1_{-{}^1/_2+u}(x) = P^1_{-{}^1/_2-u}(x), \quad -1 < x < 1$$

we can write the functions $H_n(x)$ (1.13) in the form

$$H_n(x) = C_n P^1_{-1/2+i\delta_n}(x) + D_n P^1_{-1/2+i\delta_n}(-x)$$

Since the function $P^1_{-1/2+i\delta_n}(-x)$ increases without bounds as $x \to 1$, it follows that in the case of a circular stamp the constants D_n should be assumed equal to zero so as to satisfy the condition that the contact stresses are bounded when $x \to 1$. Formula (1.12) and the second relation of the dual equation (1.5) together determine the function $q(x)$ for $x \in [-1,1]$ with the accuracy of up to the enumerable set of constants C_n and D_n. The coefficients B_n (1.6) can be determined with the same accuracy by utilizing the property of orthogonality (1.8) of the functions $y(u_k, x)$. We obtain

$$B_k = \left\{(1-x^2)\left[y(u_k, x)\left(\frac{dP_1^1(x)/dx}{(u_k^2-u_1^2)K(u_1)} + \sum_{n=1}^{\infty}\frac{H_n'(x)}{u_k^2+\delta_n^2}\right) - y'(u_k, x)\left(\frac{P_1^1(x)}{(u_k^2-u_1^2)K(u_1)} + \sum_{n=1}^{\infty}\frac{H_n(x)}{u_k^2+\delta_n^2}\right)\right]\right\}_{x=b}^{x=c} \tag{1.14}$$

In deriving (1.14) we made use of the relation

$$\int_b^c y(v, x)\, y(w, x)\, dx = \left\{\frac{1-x^2}{v^2-w^2}[y(v, x)\, y'(w, x) - y'(v, x)\, y(w, x)]\right\}_{x=b}^{x=c} \tag{1.15}$$

which holds for any distinct $(v \neq w)$ solutions $y(v, x)$ and $y(w, x)$ (i.e. for any functions $P^1_{-1/2+v}(x)$, $Q_{-1/2+v}(x)$ and $P^1_{-1/2+w}(x)$, $Q^1_{-1/2+w}(x)$ when $v \neq w$) of the equation (1.7).

2. To find the constants C_n and D_n we use the first relation of the dual equation (1.5). We note that the function

$$y^\circ(u_1, x) = \begin{cases} P_1^1(x), & b \leqslant x \leqslant c \\ 0, & -1 \leqslant x < b, \quad c < x \leqslant 1 \end{cases}$$

can be written, with (1.15) taken into account, in the form of the series

$$y^\circ(u_1, x) = \sum_{k=1}^{\infty} y(u_k, x)\left\{\frac{1-x^2}{u_k^2-u_1^2}\left[y(u_k, x)\frac{d}{dx}P_1^1(x) - y'(u_k, x)\, P_1^1(x)\right]\right\}_{x=b}^{x=c} \tag{2.1}$$

Substituting the coefficients B_k (1.14) and the function $y^\circ(u_1, x)$, written in the form of a series (2.1) ($P_1^1(x) = y^\circ(u_1, x)$ when $b \leqslant x \leqslant c$) into the first relation of the dual equation (1.5) and remembering that $K(i, \delta_n) = 0$, we obtain

$$\left\{(1-d^2)\left[K^{-1}(u_1)(P_1^1(d)\, S_d(-iu_1, x) - T_d(-iu_1, x)\, dP_1^1(d)/dd) + \sum_{n=1}^{\infty}(H_n(d)\, S_d(\delta_n, x) - H'(d)\, T_d(\delta_n, x))\right]\right\}_{d=b}^{d=c} = 0 \tag{2.2}$$

$$T_d(\varkappa, x) = \sum_{k=1}^{\infty}\frac{K(u_k) - K(i\varkappa)}{u_k^2+\varkappa^2}\, y(u_k, d)\, y(u_k, x)$$

$$S_d(\varkappa, x) = \sum_{k=1}^{\infty}\frac{K(u_k) - K(i\varkappa)}{u_k^2+\varkappa^2}\, y'(u_k, d)\, y(u_k, x) \tag{2.3}$$

We write the meromorphic function $K(u)$ (1.6) in the form of the series

$$K(u) = \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{g_m\gamma_m}{u^2+\gamma_m^2}, \quad g_m = \frac{i\pi}{[K^{-1}(i\gamma_m)]'} \tag{2.4}$$

Taking (2.4) into account, we can write the expressions (2.3) in the form

$$T_d(\varkappa, x) = -\frac{2}{\pi}\sum_{m=1}^{\infty}\frac{g_m\gamma_m}{\gamma_m^2-\varkappa^2}\rho_d(\gamma_m, x) \tag{2.5}$$

$$S_d(\varkappa, x) = -\frac{2}{\pi}\sum_{m=1}^{\infty}\frac{g_m\gamma_m}{\gamma_m^2-\varkappa^2}\sigma_d(\gamma_m, x)$$

$$\rho_d(\gamma_m, x) = \sum_{k=1}^{\infty}\frac{y(u_k, d)\, y(u_k, x)}{u_k^2+\gamma_m^2} \tag{2.6}$$

$$\sigma_d(\gamma_m, x) = \sum_{k=1}^{\infty}\frac{y'(u_k, d)\, y(u_k, x)}{u_k^2+\gamma_m^2}$$

The series (2.6) can be summed using the relation /7/

$$\sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{u-k}-\frac{1}{u+k+1}\right)P_k{}^1(x_1)\,P_k{}^{-1}(x_2)= \tag{2.7}$$
$$\frac{\pi}{\sin(\pi u)}\,P_u{}^1(x_1)\,P_u{}^{-1}(x_2)$$
$$(x_1=\cos\alpha_1,\ x_2=\cos\alpha_2,\ 0<\alpha_1<\pi,\ 0<\alpha_2<\pi,$$
$$\alpha_1+\alpha_2<\pi)$$

Performing the manipulations, we obtain

$$\rho_b(\gamma_m,x)=\lambda_m P^1_{-1/2+i\gamma_m}(-b)\,P^1_{-1/2+i\gamma_m}(x) \tag{2.8}$$
$$\rho_c(\gamma_m,x)=\lambda_m P^1_{-1/2+i\gamma_m}(c)\,P^1_{-1/2+i\gamma_m}(-x)$$
$$\sigma_b(\gamma_m,x)=\lambda_m\, dP^1_{-1/2+i\gamma_m}(-b)\,/\,db\,P^1_{-1/2+i\gamma_m}(x)$$
$$\sigma_c(\gamma_m,x)=\lambda_m\, dP^1_{-1/2+i\gamma_m}(c)\,/\,dc\,P^1_{-1/2+i\gamma_m}(-x)$$
$$\lambda_m=\pi\,[2(\gamma_m{}^2+{}^1/_4)\,\mathrm{ch}\,(\pi\gamma_m)]^{-1}$$

In this manner, we have written the functions $\rho_b(\gamma_m,x)$, $\rho_c(\gamma_m,x)$, $\sigma_b(\gamma_m,x)$ and $\sigma_c(\gamma_m,x)$ in the form of linear combinations of the functions $P^1_{-1/2+i\gamma_m}(x)$ and $P^1_{-1/2+i\gamma_m}(-x)$.

Substituting the expressions (2.5) into (2.2) with (2.8) taken into account, and equating to zero the sums of the coefficients accompanying like functions $P^1_{-1/2+i\gamma_m}(x)$ and $P^1_{-1/2+i\gamma_m}(-x)$ $(m=1,\ 2,\ \ldots)$, we obtain infinite systems of linear algebraic equations for determining the constants C_n and D_n. We write these equations in matrix form as follows:

$$BX+B_0Y=D,\quad C_0X+CY=E \tag{2.9}$$

Here $B=(b_{mn})$, $B_0=(b_{mn}{}^\circ)$, $C=(c_{mn})$, $C_0=(c_{mn}{}^\circ)$ represent known matrices of infinite order, $D=(d_m)$, $E=(e_m)$ are known column matrices (matrices of the order $\infty\times 1$), $X=(x_n)$, $Y=(y_n)$ represent unknown column matrix, and we have

$$b_{mn}=\xi_{mn}(b)\left[\frac{dP^1_{-1/2+i\gamma_m}(-b)\,/\,db}{P^1_{-1/2+i\gamma_m}(-b)}-\frac{dP^1_{-1/2+i\delta_n}(b)\,/\,db}{P^1_{-1/2+i\delta_n}(b)}\right] \tag{2.10}$$
$$b_{mn}=\frac{\xi_{mn}(b)}{P^1_{-1/2+i\delta_n}(-c)}\left[\frac{P^1_{-1/2+i\delta_n}(-b)}{P^1_{-1/2+i\gamma_m}(-b)}\,\frac{dP^1_{-1/2+i\gamma_m}(-b)}{db}-\frac{dP^1_{-1/2+i\delta_n}(-b)}{db}\right]$$
$$c_{mn}=\xi_{mn}(c)\left[\frac{dP^1_{-1/2+i\delta_n}(-c)\,/\,dc}{P^1_{-1/2+i\delta_n}(-c)}-\frac{dP^1_{-1/2+i\gamma_m}(c)\,/\,dc}{P^1_{-1/2+i\gamma_m}(c)}\right]$$
$$\overset{\circ}{c}_{mn}=\frac{\xi_{mn}(c)}{P^1_{-1/2+i\delta_n}(b)}\left[\frac{dP^1_{-1/2+i\delta_n}(c)}{dc}-\frac{P^1_{-1/2+i\delta_n}(c)}{P^1_{-1/2+i\gamma_m}(c)}\,\frac{dP^1_{-1/2+i\gamma_m}(c)}{dc}\right]$$
$$d_m=\eta_m(b)\left[\frac{dP_1{}^1(b)}{db}-\frac{P_1{}^1(b)}{P^1_{-1/2+i\gamma_m}(-b)}\,\frac{dP^1_{-1/2+i\gamma_m}(-b)}{db}\right]$$
$$e_m=\eta_m(c)\left[\frac{P_1{}^1(c)}{P^1_{-1/2+i\gamma_m}(c)}\,\frac{dP^1_{-1/2+i\gamma_m}(c)}{dc}-\frac{dP_1{}^1(c)}{dc}\right]$$
$$\xi_{mn}(d)=\frac{\sqrt{1-d^2}}{\gamma_m{}^2-\delta_n{}^2},\quad \eta_m(d)=\frac{\sqrt{1-d^2}}{(\gamma_m{}^2+u_1{}^2)K(u_1)}$$
$$x_n=C_nP^1_{-1/2+i\delta_n}(b),\quad y_n=D_nP^1_{-1/2+i\delta_n}(-c)$$

In the case of a circular stamp, we obtain the following matrix equation for determining the constants C_n :

$$BX=D \tag{2.11}$$

3. Using the asymptotic expansions /7/ of the Legendre functions, we easily obtain the following expression for large values of σ :

$$\frac{d}{dx}P^1_{-1/2+i\sigma}(\pm x)=\mp\frac{\sigma}{\sqrt{1-x^2}}P^1_{-1/2+i\sigma}(\pm x)\,[1+O(\sigma^{-1})] \tag{3.1}$$

Taking into account the relations (3.1), we note that when $\gamma_m\to\infty$, $\delta_n\to\infty$ and the values of x are fixed, then the elements $b_{mn}{}^\circ$ and $c_{mn}{}^\circ$ of the matrices B_0 and C_0 tend to zero, while the elements b_{mn} and c_{mn} (2.10) of the matrices B and C tend to the corresponding elements of the matrix $T=(\tau_{mn})$

$$\tau_{mn}=(\gamma_m-\delta_n)^{-1} \tag{3.2}$$

The elements of the matrix T^{-1} which is a two-sided inverse of T, are given by the formula /2,8/

$$t_{nm} = \{K_+'(-i\delta_n)[K_-^{-1}(i\gamma_m)]'(\delta_n - \gamma_m)\}^{-1}$$

$$K_+(u) = K_-(u) = \sqrt{A}\prod_{k=1}^{\infty}\left(\frac{u}{i\delta_k}+1\right)\left(\frac{u}{i\gamma_k}+1\right)^{-1} \tag{3.3}$$

The matrix T^{-1} (3.3) can be used to regularize the system of equations (2.9) and (2.11) /8/. We obtain the following infinite systems of linear algebraic equations: we have

$$X + T^{-1}(B - T)X = T^{-1}D \tag{3.4}$$

in the case of a circular stamp, and

$$X + T^{-1}(B - T)X + T^{-1}B_0Y = T^{-1}D \tag{3.5}$$
$$Y + T^{-1}(C - T)Y + T^{-1}C_0X = T^{-1}D$$

in the case of an annular stamp.

Solutions of the systems of equations (3.4) and (3.5) can be constructed /8/ using the method of consecutive approximations. Having found the coefficients of x_n and y_n, we obtain the contact stresses in accordance with (1.10), (1.12) and (2.10), from the formula

$$\tau_{t\varphi}(0, x) = G\varepsilon\left\{\frac{3R_0^3\sqrt{1-x^2}}{R^3 - R_0^3} + \sum_{n=1}^{\infty}\left[\frac{x_n}{P^1_{-1/2+i\delta_n}(b)}P^1_{-1/2+i\delta_n}(x) + \frac{y_n}{P^1_{-1/2+i\delta_n}(-c)}P^1_{-1/2+i\delta_n}(-x)\right]\right\} \tag{3.6}$$

The magnitude of the torsional moment applied to the stamp is given, with (1.15) taken into account, by

$$\frac{M}{2\pi R^3 G\varepsilon}\; \frac{[3(c-b) - c^3 + b^3]R_0^3}{R^3 - R_0^3} + \sum_{n=1}^{\infty}\left\{\frac{x_n[F_n(c) - F_n(b)]}{P^1_{-1/2+i\delta_n}(b)} - \frac{y_n[F_n(-c) - F_n(-b)]}{P^1_{-1/2+i\delta_n}(-c)}\right\} \tag{3.7}$$

$$F_n(x) = \frac{1-x^2}{u_1^2 - \delta_n^2}\left[P_1^1(x)\frac{d}{dx}P^1_{-1/2+i\delta_n}(x) - P^1_{-1/2+i\delta_n}(x)\frac{d}{dx}P_1^1(x)\right]$$

If the relative thickness of the spherical layer is small, we can utilize the asymptotic expansion /7/ of the function $P^1_{-1/2+i\delta_n}(x)$ for large values of δ_n ($\delta_n > 32n$ for $|R_0 - R|R^{-1} \leqslant 0.1$), to obtain

$$\frac{P^1_{-1/2+i\delta_n}(x)}{P^1_{-1/2+i\delta_n}(b)} = \left(\frac{1-b^2}{1-x^2}\right)^{1/4}\left[\exp(-\delta_n(\theta_2 - \theta)) + O\left(\frac{\exp(-\delta_n\theta_2)}{\delta_n}\right)\right] \tag{3.8}$$

$$\frac{P^1_{-1/2+i\delta_n}(-x)}{P^1_{-1/2+i\delta_n}(-c)} = \left(\frac{1-c^2}{1-x^2}\right)^{1/4}\left[\exp(-(\theta - \theta_1)\delta_n) + O\left(\frac{\exp(-\delta_n(\pi - \theta_1))}{\delta_n}\right)\right]$$

The solutions /8/

$$X_0 = (x_n^\circ), \quad Y_0 = (y_n^\circ)$$

$$\frac{x_n^\circ}{\sqrt{1-b^2}} = \frac{y_n^\circ}{\sqrt{1-c^2}} = \frac{3R_0^3}{R^3 - R_0^3}\,\frac{(2n-1)!!}{(2n)!!} \tag{3.9}$$

of the system of equations

$$T_0X_0 = D_0, \quad T_0Y_0 = E_0$$

$$T_0 = (\mathring{\tau}_{mn}), \quad D_0 = (d_m^\circ), \quad E_0 = (e_m^\circ), \quad \mathring{\tau}_{mn} = (\gamma_m^\circ - \delta_n)^{-1}$$

$$d_m^\circ = \frac{3R_0^3\sqrt{1-b^2}}{(R_0^3 - R^3)\gamma_m^\circ}, \quad e_m^\circ = \frac{3R_0^3\sqrt{1-c^2}}{(R_0^3 - R^3)\gamma_m^\circ}, \quad \gamma_m^\circ = \frac{(2m-1)\pi}{2|a|}$$

can be used as the principal terms of the asymptotics (when the relative thickness $|R_0 - R|R^{-1}$ of the spherical layer is small) of the solution of (2.9).

Carrying out the summation of the series in (3.6) with (3.8) and (3.9) taken into account, we obtain the following formula for approximate determination of the contact stresses for a spherical layer of small relative thickness:

$$\tau_{r\varphi}(R, \theta) = \frac{3R_0^3G\varepsilon\sin\theta}{R^3 - R_0^3}\left\{1 - \left(\frac{\sin\theta_2}{\sin\theta}\right)^{3/2}\left[1 - \left(1 - \exp\left(-\frac{\pi(\theta_2 - \theta)}{|a|}\right)\right)^{-1/2}\right] - \left(\frac{\sin\theta_1}{\sin\theta}\right)^{3/2}\left[1 - \left(1 - \exp\left(-\frac{\pi(\theta - \theta_1)}{|a|}\right)\right)^{-1/2}\right]\right\} \tag{3.10}$$

The dependence of the torsional moment M (3.7) applied to the stamp, on the angle of rotation ε, is given, with (3.1), (3.8) and (3.9) taken into account, by the relation

$$M = 6\pi R^3 R_0^3 G\varepsilon (R^3 - R_0^3)^{-1} [\cos\theta_1 - \cos\theta_2 - {}^1/_3 (\cos^3\theta_1 - \cos^3\theta_2) + (\sin^3\theta_1 + \sin^3\theta_2)|a|\pi^{-1} \ln 4] \tag{3.11}$$

We note that according to (3.10) the contact stresses increase without bounds on approaching the boundaries of the stamp. At the boundaries of the stamp (when $\theta = \theta_1$ and $\theta = \theta_2$) the contact stresses have root-type singularities

$$\lim_{\theta\to\theta_1} \tau_{r\varphi}(R, \theta) \sqrt{\theta\theta_1^{-1} - 1} = f(\theta_1)$$

$$\lim_{\theta\to\theta_2} \tau_{r\varphi}(R, \theta) \sqrt{1 - \theta\theta_2^{-1}} = f(\theta_2)$$

$$f(\theta) = \frac{3R_0^3 G\varepsilon}{R^3 - R_0^3} \sqrt{\frac{|a|}{\pi\theta}} \sin\theta$$

If the spherical layer is rotated by a circular spherical stamp, then we must put in (3.10) and (3.11) $\theta_1 = 0$.

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